LOGIC IN NUMBERS.

Logic is the science of consistency. Given a set of propositions, the fundamental problem of logic is to determine whether the propositions can be true together. It is possible to reduce this fundamental problem to a purely mathematical form and to transfer the problem from the domain of philosophy to the domain of mathematics. The sytem of Boole and other systems derived from his, employ mathematical symbols in logical investigations, but meanings are attributed to the symbols that prevent the application of ordinary mathematical processes and it is impossible to proceed beyond cases of extreme simplicity. By the method here outlined the problem of the logician, however intricate, may be expressed as a purely mathematical problem, in the statement of which + means *plus* and − means *minus* and $2+2=4$. The propositions may be expressed as a set of whole numbers and the consistency of the propositions depends upon whether the numbers can be divided into two groups such that the sum of the numbers placed in one group is equal to the sum of the numbers placed in the other group. If the two equal groups can be formed, the propositions are consistent. If it is impossible to form the two equal groups, then the propositions are inconsistent, that is to say, the propositions cannot all be true. Whether the two equal groups can be formed from the numbers arising from the proposition, is a question for the mathematician to answer. It will be necessary to define a few terms.

The sum of the coefficients of a polynomial, all being regarded as positive, is the *weight* of the polynomial. Half the weight is the *semi-weight*. If the polynomial can be made equal to 0 by making each variable either $+1$ or -1, the polynomial is a *balance*. If the variables are written down (without coefficients) and those that are made $+1$ in order to make the polynomial vanish are written with the + sign, and those that are made -1 in order to make the polynomial vanish are written with the − sign, the expression is a *solution* of the balance.

Thus $2a+3b+7c-5d-e$ is a balance, for if a, c, d and e are each $+1$ and b is -1, the polynomial becomes $2-3+7-5-1$, which is equal to 0. $a-b+c+d+e$ is a solution of the balance. The weight of the balance is $2+3+7+5+1$ or 18.

Note that if it is possible to divide the coefficients of a polynomial, all being regarded as positive, into two groups such that

the sum of the coefficients placed in either group is the semi-weight of the polynomial, the polynomial is a balance. If the variables belonging to the coefficients placed in one of such groups are written with the signs they have in the balance, and the variables belonging to the coefficients placed in the other group are written with signs contrary to the signs they have in the balance, the variables form a solution.

If letters that do not appear in a balance are added to or are subtracted from a solution, the expression is still a solution. Such additional letters may be regarded as being in the balance with the coefficient 0.

Thus $a-b+c+d+e+f-g$ is a solution of the balance, $2a+3b+7c-5d-e$.

If two or more balances have a common solution they are said to be *consistent* and to form a consistent system, and the common solutions are solutions of the system. But if there is no common solution the balances are *inconsistent* and form an inconsistent system.

Thus a system composed of the balances,

$$\begin{array}{l} 13m+4a+10b+3c+9d+n+3p+9q \\ 4m+3a \qquad\quad +\ c-2d \qquad\qquad\qquad +r+3s \end{array}$$

is consistent, for $m+a-b-c-d-n-p+q+r-s$ is a solution of both balances. But a system composed of the same two balances and the balance,

$$m \quad -b \quad -c \qquad\qquad\qquad +t$$

is inconsistent, for these three balances have no common solution.

If any letter must have the same sign as another letter in every solution of a balance or system of balances, the two letters are said to be *identical,* and two letters that have different signs in every solution are said to be *contradictory.*

Thus, in the system of two consistent balances mentioned in the preceding paragraph, the letters b and c are identical and m and b are contradictory, and so are m and c.

If a polynomial is constructed such that all solutions it may have are solutions of a system of balances, and such that all solutions there may be of the system are solutions of the polynomial, the polynomial is a *summary* of the system.

Thus $4m+a+b+n+3c+3d+3p$ is a summary of the two balances, $m+a+b+n$ and $m+c+d+p$.

The *sum* (and the difference) of any two given consistent balances is a balance consistent with them.

If there are two balances, a summary may be obtained by adding them together (or by subtracting one from the other), after multiplying one of them by any number that is greater than half the weight of the other.

Thus, if there are two balances,

$$\begin{array}{llllll} 2m & +2a & +b & +c & +n & +p \\ m & & -b & -c & & +t \end{array}$$

multiply the first by any number greater than 2 (which is half the weight of the second balance), say 3, and then add the second. We obtain,

$$7m+6a+2b+2c+3n+3p+t,$$

which is a summary of the two balances.

We may obtain a summary of any system of balances by adding the balances together (or by adding some and subtracting others) after multiplying the first of the balances by 1 and each of the others by successive powers of any number greater than half the weight of the balance that has the greatest weight.

Thus, if we have the system,

$$\begin{array}{l} 4m+a+c+e+g+i+n \\ 3m+a+c+e+g+i \quad +b+d+f+h \\ 4m \qquad\quad +e \qquad\qquad +b+d+f+h+p, \end{array}$$

a summary may be obtained by adding the balances together after multiplying the first by 1, the second by any number greater than 6 (which is half the weight of the second balance, which has the greatest weight), say 7, and the third by 49 (the square of 7). We thus obtain

$$221m+8a+8c+57e+8g+8i+n+56b+56d+56f+56h+49p.$$

This expression is a summary of the given system of three balances. (This summary is not a balance; hence it may be inferred that the system from which it is derived is inconsistent.)

A summary of a system of balances may be at once obtained thus: Arrange the balances so that the several letters, as they occur in the different balances, are each in a separate column. (When a letter that appears in the system does not appear in any particular balance, it may be supposed to be inserted in that balance with the coefficient 0.) The coefficient of any letter in the summary is a number obtained by writing the coefficients of the letter in the order in which they appear in the column containing that letter,

commencing with the coefficient in the first balance as standing in the unit's place, the number so obtained being regarded as expressed in any scale whose radix is greater than half the weight of the balance that has the greatest weight.

Applying this method to the example in the preceding paragraph, the coefficient of m is 434; that of a is 11; that of n is 1; that of b is 110; that of p is 100; etc.; all read in any scale greater than 6. The summary may be written,

$$434m + 11a + 11c + 111e + 11g + 11i + n + 110b + 110d + 110f + 110h + 100p.$$

If any of the coefficients of the balances have negative signs the same rule may be applied for obtaining the coefficients of a summary, but in the number expressing the coefficient of such letter in the summary, negative numerals are used to correspond with the negative coefficients of the letter in the balances.

Thus, in the system

$$\begin{array}{llllllll} 3m & +2a & +n & +p & +t & & & \\ 3m & & & & +t & +2d & +q & +r \\ m & -a & & & & -d & & +s \end{array}$$

a summary is

$$133m + (-1)02a + n + p + 11t + (-1)20d + 10q + 10r + 100s$$

the numbers being in any scale greater than 4. The summary may be written in the scale of 5 thus,

$$133m - 43a + n + p + 10t - 30d + 10q + 10r + 100s.$$

If a summary is a balance, the system from which it is derived must be consistent; if, however, a summary is not a balance, the system from which it is derived must be an inconsistent system. The consistency, therefore, of a system of balances may be tested by reading off a summary and determining whether the summary is a balance.

Universal propositions may be expressed as balances. A balance represents a universal proposition, if all its solutions represent all cases that are conceivable, if the proposition be true. In the solution of a balance, or system of balances, let m (the first letter of *mundus*) represent something that is conceivable as existing in the universe of discourse; let $m+a$ (or $-m-a$) represent something that is conceivable as existing and as having an attribute denoted by a; let $m-a$ (or $-m+a$) represent something that is conceivable as

existing and as not having the attribute denoted by a; similarly, let $m+a-b$ (or $-m-a+b$) represent something that is conceivable as existing and as having the attribute denoted by a and as not having the attribute denoted by b; generally, let any solution represent something that is conceivable as having the attributes denoted by the letters with one sign and as not having the attributes denoted by the letters with the other sign.

The balance $m+a+b+n$ expresses the universal proposition "No a is b," for all its solutions represent all cases that are possible, if the proposition be true. In every solution one of the letters, m, a and b, has a sign different from that of the other two, which is exactly what is required by the proposition.

The balance $m+a-b+p$ expresses the universal affirmative, All a is b." In every solution, if a has the same sign as m, b has also the same sign as m; but if m and a have different signs, then b may be + or –.

Sometimes it may be convenient to express a proposition by a system of balances instead of by a single balance. Thus "All a is b" may be expressed by the two balances.

$$\begin{array}{ll} m+a+q+r & \\ & q+r+b+s \end{array}$$

Any universal proposition may be stated as a balance or system of balances. The following are given as illustrations:

Whatever is conceivable is a: $m-a$.

Nothing can be a: $m+a$.

a and b are identical: $a-b$.

a and b are contradictory: $a+b$.

a is neither b nor c: $\begin{array}{ll} m+a+b+q & \\ m+a & +c+r. \end{array}$

a is either b or c, or both b and c: $\begin{array}{ll} m+a+p+q & \\ p & +b+c+r. \end{array}$

a is either b or c, but not both: $\begin{array}{ll} m+a-b-c+s+t & \\ m+a+b+c & +u+v. \end{array}$

Of the three terms, a, b and c, two, at least, are absent: $2m+a+b+c+n$.

Everything has at least two of the attributes denoted by a, b and c: $2m-a-b-c+p$.

Of n things, a_1, $a_2 \ldots . a_n$, p at least are present and q at least are absent: $(q-p)m+a_1+a_2 \ldots +a_n+r_1+r_2+\ldots +r_{n-p-q}$.

To test the consistency of universal propositions, therefore, they

may be expressed as a system of balances, a summary may be read off, and whether the propositions are or are not consistent depends upon the purely mathematical question whether the summary is a balance.

Universal propositions express rules that must be observed in every solution of a system of balances expressing the propositions. A particular proposition expresses a rule that must be observed in at least one solution of the system expressing universal propositions. If a set of universal propositions and a particular proposition is given, to test their consistency a summary may be obtained of the universal propositions, and then certain variables may be given + or − signs in accordance with the particular proposition; then, if the summary is a balance, the particular proposition is consistent with the universal propositions; otherwise, it is not. Thus, if universal propositions and the particular proposition, "Some *a* is *b*" are given, there must be a solution of the summary of the universal propositions in which *m, a* and *b* have the same sign. If in the summary *m, a* and *b* are made $+1$, and the summary is still a balance, the propositions are consistent. If there are several particular propositions, the summary should be tested as to each one separately. It is to be observed that there is no implication that any solution of the summary must comply with more than one of the particular propositions.

The method here outlined is a general method of converting logical problems into a mathematical form. It is possible, however, to solve many problems by manipulating balances otherwise and there are a number of important theorems in regard to these expressions, but it would be beyond the purpose of this paper to enter upon a discussion of them. The following problems may serve to illustrate a method of obtaining solutions of balances.

If five chess queens are placed on a board containing 25 cells arranged in the form of a square, so that no two queens attack each other, prove that neither of the diagonals of the square can be without a queen.

Let (x, y) represent a cell which is the xth from the left and the yth from the bottom of the board. Of the cells, (1, 5), (2, 4), (3, 3), (4, 2) and (5, 1), forming a diagonal, four at least are vacant. Hence the balance,

$$4m+(1,5)+(2,4)+(3,3)+(4,2)+(5,1)+p_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathrm{I})$$

Similarly fourteen other balances may be formed, each of which contains $4m$ and p with a different suffix, and also five cells indicated

1 8. 12.	3 9 12.	7.. 12.	3 10. 12.	2 11. 12.
4 8. 13.	1 9. 13.	3 4 6 13.	2 10. 13.	6 11. 13.
7.. 8.	3 4 5 9.	1 2 7..	3 5 6 10.	7.. 11.
4 8. 14.	2 9. 14.	4 5 6 14.	1 10. 14.	6 11. 14.
2 8. 15.	5 9. 15.	7.. 15.	5 10. 15.	1 11. 15.

in the diagram by a number corresponding with the suffix of p to be used with them. Thus, one of such balances is

$$4m+(2,5)+(4,5)+(3,4)+(2,3)+(4,3)+p_3.$$

If the fifteen balances are added together, after multiplying by 3 the balance that has p_7 and by 2 the balances that have p_8, p_9, p_{10}, p_{11}, p_{12}, p_{13}, p_{14}, and p_{15}, we obtain,

$$100m+5S+p_1+p_2+p_3+p_4+p_5+p_6+3p_7+2(p_8+p_9+p_{10} +p_{11}+p_{12}+p_{13}+p_{14}+p_{15}) \quad \ldots\ldots\ldots\ldots \quad \text{(II)}$$

where S represents the 25 cells of the board.

Since 5 cells are occupied and 20 are vacant, we have the balance

$$15m+S \quad \ldots\ldots\ldots\ldots \quad \text{(III)}$$

Subtracting 5 times (III) from (II) we get

$$25m+p_1+p_2+p_3+p_4+p_5+p_6+3p_7+2(p_8+p_9+p_{10}+p_{11}+p_{12} p_{13}+p_{14}+p_{15}).$$

The weight of this last balance is 50, and the coefficient of m is 25. Hence m is the contradictory of each of the p's. Therefore,

$$m+p_1 \quad \ldots\ldots\ldots\ldots \quad \text{(IV)}$$

is a balance. Subtracting (IV) from (I) we get,

$$3m+(1,5)+(2,4)+(3,3)+(4,2)+(5,1).$$

In every solution of this balance one of the cells has the same sign as m, and the other four cells have the opposite sign. Hence, one of these five cells, forming that diagonal, must be occupied and the other four must be vacant.

Similarly it may be shown that one of the cells forming the other diagonal must be occupied.

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